

AN EXPLICIT $\bar{\partial}$ -INTEGRATION FORMULA FOR WEIGHTED HOMOGENEOUS VARIETIES

J. RUPPENTHAL AND E. S. ZERON

ABSTRACT. Let Σ be a weighted homogeneous (singular) subvariety of \mathbb{C}^n . The main objective of this paper is to present an explicit formula for solving the $\bar{\partial}$ -equation $\lambda = \bar{\partial}g$ on the regular part of Σ , where λ is a $\bar{\partial}$ -closed $(0, 1)$ -form with compact support. This formula will then be used to give Hölder estimates for the solution in case Σ is homogeneous (a cone) with an isolated singularity. Finally, a slight modification of our formula also gives an L^2 -bounded solution operator in case Σ is pure d -dimensional and homogeneous.

1. INTRODUCTION

As it is well known, solving the $\bar{\partial}$ -equation forms a main part of complex analysis, but also has deep consequences on algebraic geometry, partial differential equations and other areas. In general, it is not easy to solve the $\bar{\partial}$ -equation. The existence of solutions depends mainly on the geometry of the variety on which the equation is considered. There is a vast literature about this subject on smooth manifolds, both in books and papers [10, 11], but the theory on singular varieties has been developed only recently.

Let Σ be a singular subvariety of the space \mathbb{C}^n , and λ be a $\bar{\partial}$ -closed differential form well defined on the regular part of Σ . Fornæss, Gavosto and Ruppenthal have produced a general technique for solving the $\bar{\partial}$ -equation $\lambda = \bar{\partial}g$ on the regular part of Σ ; see [9, 6] and [12]. On the other hand, Acosta, Solís and Zeron have proposed an alternative technique for solving the $\bar{\partial}$ -equation when Σ is a quotient variety; see [1, 2] and [14].

Date: January 10, 2008.

2000 Mathematics Subject Classification. 32F20, 32W05, 35N15.

Key words and phrases. Cauchy-Riemann equations, Hölder estimates, L^2 -estimates, resolution of singularities.

The first author was supported by a fellowship within the Postdoc-Programme of the German Academic Exchange Service (DAAD). The second author was supported by Cinvestav(Mexico) and Conacyt-SNI(Mexico).

In both cases, the main strategy is to transfer the problem to some non-singular complex space, to solve the $\bar{\partial}$ -equation in this well-known situation, and to carry over the solution to the singular variety. The main objective of this paper is to present and analyze an explicit formula for calculating solutions g to the $\bar{\partial}$ -equation $\lambda = \bar{\partial}g$ on the regular part of the original variety Σ , where Σ is a weighted homogeneous variety and λ is a $\bar{\partial}$ -closed $(0,1)$ -differential form with compact support. We analyze the weighted homogeneous varieties, for they are a main model for classifying the singular subvarieties of \mathbb{C}^n . A detailed analysis of the weighted homogeneous varieties is done in Chapter 2–§4 and Appendix B of [4].

Definition 1. *Let $\beta \in \mathbb{Z}^n$ be a fixed integer vector with strictly positive entries $\beta_k \geq 1$. A polynomial $Q(z)$ holomorphic on \mathbb{C}^n is said to be **weighted homogeneous** of degree $d \geq 1$ with respect to β if the following equality holds for all $s \in \mathbb{C}$ and $z \in \mathbb{C}^n$:*

$$(1) \quad Q(s^\beta * z) = s^d Q(z), \quad \text{with the action:}$$

$$(2) \quad s^\beta * (z_1, z_2, \dots, z_n) := (s^{\beta_1} z_1, s^{\beta_2} z_2, \dots, s^{\beta_n} z_n).$$

*Besides, an algebraic subvariety Σ in \mathbb{C}^n is said to be **weighted homogeneous** with respect to β whenever Σ is the zero locus of a finite number of weighted homogeneous polynomials $Q_k(z)$ of (maybe different) degrees $d_k \geq 1$, but all of them with respect to the same fixed vector β .*

Let $\Sigma \subset \mathbb{C}^n$ be any subvariety. We use the following notation along this paper. The regular part $\Sigma^* = \Sigma_{reg}$ is the complex manifold composed by all the regular points of Σ , and it is always endowed with the induced metric; so that Σ^* is a Hermitian submanifold in \mathbb{C}^n with corresponding volume element dV_Σ and induced norm $|\cdot|_\Sigma$ on the Grassmannian $\Lambda T^* \Sigma^*$. Thus, any Borel-measurable $(0,1)$ -form λ on Σ^* admits a representation $\lambda = \sum_k f_k d\bar{z}_k$, where the coefficients f_k are Borel-measurable functions on Σ^* which satisfy the inequality $|f_k(w)| \leq |\lambda(w)|_\Sigma$ for all points $w \in \Sigma^*$ and indexes $1 \leq k \leq n$. Notice that such a representation is by no means unique. We refer to Lemma 2.2.1 in [12] for a more detailed treatment of that point. We also introduce the L^2 -norm of a measurable (p,q) -form \aleph on an open set $U \subset \Sigma^*$ via the formula:

$$\|\aleph\|_{L^2_{p,q}(U)} := \left(\int_U |\aleph|_\Sigma^2 dV_\Sigma \right)^{1/2}.$$

We can now present the main result of this paper. We assume that the $\bar{\partial}$ -differentials are calculated in the sense of distributions, for we work with Borel-measurable functions.

Theorem 2 (Main). *Let Σ be a weighted homogeneous subvariety of \mathbb{C}^n with respect to a given vector $\beta \in \mathbb{Z}^n$, where $n \geq 2$ and all entries $\beta_k \geq 1$. Consider a $(0,1)$ -form λ given by $\sum_k f_k d\bar{z}_k$, where the coefficients f_k are all Borel-measurable functions in Σ , and z_1, \dots, z_n are the Cartesian coordinates of \mathbb{C}^n . The following function is well defined for all $z \in \Sigma$ whenever the form λ is bounded and has compact support in Σ :*

$$(3) \quad g(z) := \sum_{k=1}^n \frac{\beta_k}{2\pi i} \int_{w \in \mathbb{C}} f_k(w^\beta * z) \frac{\overline{(w^{\beta_k} z_k)} dw \wedge d\bar{w}}{\bar{w} (w-1)}.$$

Besides, the function g is a solution of the $\bar{\partial}$ -equation $\lambda = \bar{\partial}g$ on the regular part of Σ , whenever λ is also $\bar{\partial}$ -closed on the regular part of Σ .

Notice that $g(0) = 0$ and that (3) can also be rewritten as follows after the change of variables $u = ws$, given $s \in \mathbb{C}$ and $z \in \Sigma$,

$$(4) \quad g(s^\beta * z) = \sum_{k=1}^n \frac{\beta_k}{2\pi i} \int_{u \in \mathbb{C}} f_k(u^\beta * z) \frac{\overline{(u^{\beta_k} z_k)} du \wedge d\bar{u}}{\bar{u} (u-s)}.$$

We shall prove Theorem 2 in Section 2 of this paper. Moreover, recalling some main principles of the proof, we also deduce anisotropic Hölder estimates for the $\bar{\partial}$ -equation in the case where Σ is a homogeneous variety with an isolated singularity at the origin. We obviously need to specify the metric on Σ : Given a pair of points z and w in Σ , we define $\text{dist}_\Sigma(z, w)$ to be the infimum of the length of all piecewise smooth curves connecting z and w inside Σ . It is clear that such curves exist in this situation, and that the length of each curve can be measured in the regular part Σ^* or the ambient space \mathbb{C}^n , but both measures coincide, for Σ^* carries the induced norm. The main result of the section 3 is the following estimate:

Theorem 3 (Hölder). *In the situation of Theorem 2, suppose that Σ is homogeneous (a cone) and has got only one isolated singularity at the origin of \mathbb{C}^n , so that each entry $\beta_k = 1$ in Definition 1. Moreover, assume that the support of the form λ is contained in a ball B_R of radius $R > 0$ and center at the origin. Then, for each parameter $0 < \theta < 1$, there exists a strictly positive constant $C_\Sigma(R, \theta)$ which does not depend on λ such that the following inequality holds for the function g given in (3) and all points z and*

w in the intersection $B_R \cap \Sigma$,

$$(5) \quad |g(z) - g(w)| \leq C_\Sigma(R, \theta) \cdot \text{dist}_\Sigma(z, w)^\theta \cdot \|\lambda\|_\infty.$$

The notation $\|\lambda\|_\infty$ stands for the essential supremum of $|\lambda(w)|_\Sigma$ on Σ , recall that λ is bounded and has compact support. Theorem 3 is proved in Section 3. Finally, similar techniques and a slight modification of equation (3), can also be used for producing a $\bar{\partial}$ -solution operator with L^2 -estimates on homogeneous subvarieties with an isolated singularity at the origin.

Theorem 4 (L^2 -Estimates). *Let Σ be a pure d -dimensional homogeneous (cone) subvariety of \mathbb{C}^n , where $n \geq 2$ and each entry $\beta_k = 1$ in Definition 1. Consider a $(0,1)$ -form λ given by $\sum_k f_k d\bar{z}_k$, where the coefficients f_k are all square integrable functions in Σ , and z_1, \dots, z_n are the Cartesian coordinates of \mathbb{C}^n . The following function is well defined for almost all $z \in \Sigma$ whenever the form λ has compact support on Σ :*

$$(6) \quad g(z) := \sum_{k=1}^n \frac{1}{2\pi i} \int_{w \in \mathbb{C}} f_k(wz) \frac{w^{d-1} \bar{z}_k dw \wedge d\bar{w}}{w-1}.$$

The function g is a solution of the $\bar{\partial}$ -equation $\lambda = \bar{\partial}g$ on the regular part of Σ , whenever λ is also $\bar{\partial}$ -closed on the regular part of Σ . Finally, assuming that the support of λ is contained in an open ball B_R of radius $R > 0$ and center in the origin, there exists a strictly positive constant $C_\Sigma(R, 2)$ which does not depend on λ and such that:

$$(7) \quad \|g\|_{L^2(\Sigma \cap B_R)} \leq C_\Sigma(R, 2) \cdot \|\lambda\|_{L^2_{0,1}(\Sigma)}.$$

We prove this theorem in Section 4 of this paper. The obstructions to solving the $\bar{\partial}$ -equation with L^2 -estimates on singular complex spaces are not completely understood in general. An L^2 -solution operator is only known for the case when Σ is a complete intersection of pure dimension ≥ 3 with isolated singularities only. This operator was built by Fornæss, Øvrelid and Vassiliadou in [8], via an extension theorem for the $\bar{\partial}$ -cohomology groups originally presented by Scheja [13]. The L^2 -results usually come with some obstructions to the solvability of the $\bar{\partial}$ -equation. For example, different situations are analyzed in the works of Diederich, Fornæss, Øvrelid and Vassiliadou; it is shown there that the $\bar{\partial}$ -equation is solvable with L^2 -estimates for all forms lying in a closed subspace of finite codimension of the vector space of all the $\bar{\partial}$ -closed L^2 -forms [3, 5, 8, 15]. Besides, in the paper [7], the $\bar{\partial}$ -equation is solved locally with some weighted L^2 -estimates for forms

which vanish to a sufficiently high order on the singular set of the given varieties.

On the other hand, we propose in Section 5 of this paper a useful technique for generalizing the estimates given in Theorems 3 and 4, so as to consider weighted homogeneous subvarieties instead of homogeneous ones.

2. PROOF OF MAIN THEOREM

Let $\{Q_k\}$ be the set of polynomials on \mathbb{C}^n which defines the algebraic variety Σ as its zero locus. The definition of weighted homogeneous varieties implies that the polynomials $Q_k(z)$ are all weighted homogeneous with respect to the same fixed vector β . Equation (1) automatically yields that every point $s^\beta * z$ lies in Σ for all $s \in \mathbb{C}$ and $z \in \Sigma$, and so each coefficient $f_k(\cdot)$ in equations (3) and (4) is well evaluated in Σ . Moreover, fixing any point $z \in \Sigma$, the given hypotheses imply that the following Borel-measurable functions are all bounded and have compact support in \mathbb{C} ,

$$w \mapsto f_k(w^\beta * z).$$

Hence, the function $g(z)$ in (3) is well defined for every $z \in \Sigma$. Notice that $g(0) = 0$, in particular. We shall prove that $g(z)$ is also a solution of the equation $\bar{\partial}g = \lambda$, when the $(0,1)$ -form λ is $\bar{\partial}$ -closed. We may suppose, without loss of generality, that the regular part of Σ does not contain the origin; see Lemma 4.3.2 in [12]. Let $\xi \neq 0$ be any fixed point in the regular part of Σ . We may also suppose by simplicity that the first entry $\xi_1 \neq 0$, and so we define the following mapping $\eta : \mathbb{C}^n \rightarrow \mathbb{C}^n$ and variety Y ,

$$(8) \quad \begin{aligned} \eta(y) &:= (y_1/\xi_1)^\beta * (\xi_1, y_2, y_3, \dots, y_n), \quad \text{for } y \in \mathbb{C}^n, \\ Y &:= \{\hat{y} \in \mathbb{C}^{n-1} : Q_k(\xi_1, \hat{y}) = 0, \forall k\}. \end{aligned}$$

The action $s^\beta * z$ was given in (2). We have that $\eta(\xi) = \xi$, and that the following identities hold for all $s \in \mathbb{C}$ and $\hat{y} \in \mathbb{C}^{n-1}$ (recall equation (1) and the fact that Σ is the zero locus of the polynomials $\{Q_k\}$):

$$(9) \quad \begin{aligned} Q_k(\eta(s, \hat{y})) &= (s/\xi_1)^{d_k} Q_k(\xi_1, \hat{y}), \quad \text{and so} \\ \eta(\mathbb{C}^* \times Y) &= \{z \in \Sigma : z_1 \neq 0\}. \end{aligned}$$

The symbol \mathbb{C}^* stands for $\mathbb{C} \setminus \{0\}$. The mapping $\eta(y)$ is locally a biholomorphism whenever the first entry $y_1 \neq 0$. Whence, the point ξ lies in the regular part of the variety $\mathbb{C} \times Y$, because $\xi = \eta(\xi)$ also lies in the regular part of Σ and $\xi_1 \neq 0$. Thus, we can find a biholomorphism π defined from an open domain U in \mathbb{C}^m onto an open set in the regular part of Y , such

that $\pi(\rho)$ is equal to (ξ_2, \dots, ξ_n) for some $\rho \in U$. Consider the following holomorphic mapping defined for all points $s \in \mathbb{C}$ and $x \in U$,

$$(10) \quad \Pi(s, x) := s^\beta * (\xi_1, \pi(x)) = \eta(s\xi_1, \pi(x)) \in \Sigma.$$

The image $\Pi(\mathbb{C} \times U)$ will be known as a **generalized cone** from now on. Notice that $\Pi(\mathbb{C}^* \times U)$ lies in the regular part of Σ , for $\pi(U)$ is contained in the regular part of Y . The mapping $\Pi(s, x)$ is locally a biholomorphism whenever $s \neq 0$, because η is also a local biholomorphism for $y_1 \neq 0$. Finally, the image $\Pi(1, \rho)$ is equal to ξ . Hence, recalling the differential form λ and the function g defined in (3), we only need to prove that the pull-back $\Pi^*\lambda$ is equal to $\bar{\partial}g(\Pi)$ inside $\mathbb{C} \times U$, in order to conclude that the $\bar{\partial}$ -equation $\lambda = \bar{\partial}g$ holds in a neighborhood of ξ . Consider the following identity obtained by applying (2) and (10) into (3), we define $\pi_1(x) \equiv \xi_1$,

$$(11) \quad g(\Pi(s, x)) = \sum_{k=1}^n \frac{\beta_k}{2\pi i} \int_{\mathbb{C}} f_k(\Pi(ws, x)) \frac{\overline{(ws)^{\beta_k} \pi_k(x)} dw \wedge d\bar{w}}{\bar{w} (w-1)}.$$

The given hypotheses on λ yield that the pull-back $\Pi^*\lambda$ is $\bar{\partial}$ -closed and bounded in $\mathbb{C}^* \times U$, and so it is also $\bar{\partial}$ -closed in $\mathbb{C} \times U$; see Lemma 4.3.2 in [12] or Lemma (2.2) in [14]. We can then use equations (2) and (10) in order to calculate $\Pi^*\lambda$ when λ is given by $\sum_k f_k dz_k$,

$$(12) \quad \begin{aligned} \Pi^*\lambda &= F_0(s, x) d\bar{s} + \sum_{j \geq 1} F_j(s, x) d\bar{x}_j, \quad \text{with} \\ F_0(s, x) &= \sum_{k=1}^n f_k(\Pi(s, x)) \beta_k \overline{s^{\beta_k-1} \pi_k(x)}. \end{aligned}$$

Recall that $\pi_1(x) \equiv \xi_1$. Equation (10) and the fact that λ has compact support on Σ also imply that the previous function $F_0(s, x)$ has compact support on every complex line $\mathbb{C} \times \{x\}$, for all $x \in U$. Whence, the following Cauchy-Pompeiu integral satisfies the $\bar{\partial}$ -equation $\Pi^*\lambda = \bar{\partial}G$ in the product $\mathbb{C} \times U$, given $s \in \mathbb{C}$ and $x \in U$,

$$(13) \quad G(s, x) := \frac{1}{2\pi i} \int_{u \in \mathbb{C}} \frac{F_0(u, x)}{u-s} du \wedge d\bar{u}.$$

Finally, equations (11) and (13) are identical, for we only need to apply the change of variables $u = sw$. Thus, the differential $\bar{\partial}g(\Pi)$ (resp. $\bar{\partial}g$) is equal to the form $\Pi^*\lambda$ (resp. λ) inside the space $\mathbb{C} \times U$ (resp. an open neighborhood of ξ); and so, the $\bar{\partial}$ -equation $\lambda = \bar{\partial}g$ holds in the regular part of Σ , because $\xi \neq 0$ was chosen in an arbitrary way in the regular part of Σ , and Lemma 4.3.2 in [12].

3. HÖLDER ESTIMATES

In this section, we will prove anisotropic Hölder estimates on the subvariety $\Sigma \subset \mathbb{C}^n$ in the particular case when Σ is homogeneous (a cone) and has got only one isolated singularity at the origin (Lemma 5). These estimates easily lead to optimal Hölder estimates on such varieties (Theorem 3). We will show later (in section 5) how we can use previous results in order to deduce Hölder estimates on weighted homogeneous varieties with an isolated singularity as well. The given hypotheses imply that $\Sigma \setminus \{0\}$ is a regular complex manifold in \mathbb{C}^n . Consider the compact **link** K obtained by intersecting Σ with the unit sphere bB of radius \sqrt{n} and center at the origin in \mathbb{C}^n . Notice that every point $\xi \in K$ has got at least one coordinate with absolute value $|\xi_k| \geq 1$. We follow the proof of Theorem 2.

Thus, given any point $\xi \in K$, we construct a generalized cone which contains it. For example, if the first entry $|\xi_1| \geq 1$, we build the subvariety Y_ξ as in (8). Then, we consider a biholomorphism π_ξ defined from an open set $U_\xi \subset \mathbb{C}^m$ into a neighborhood of (ξ_2, \dots, ξ_n) in Y_ξ , and the mapping Π_ξ defined as in (10) from $\mathbb{C} \times U_\xi$ into Σ . We also restrict the domain of Π_ξ to a smaller set $\mathbb{C} \times U_\xi''$, where: $U_\xi'' \Subset U_\xi' \Subset U_\xi$, the open set U_ξ' is smoothly bounded, and $\pi_\xi(U_\xi'')$ is a convex open neighborhood of (ξ_2, \dots, ξ_n) in Y_ξ . The generalized cone $\Pi_\xi(\mathbb{C} \times U_\xi'')$ obviously contains ξ , as we wanted. Recall that an open set V in Y_ξ is called convex whenever every pair of points in V can be joined by a geodesic which is also contained in V . We proceed in a similar way when any other entry $|\xi_k| \geq 1$.

Now then, since the link K is compact, we may choose finitely many (let us say N) points ξ^1, \dots, ξ^N in K such that K itself is covered by their associated generalized cones $C_j := \Pi_{\xi^j}(\mathbb{C} \times U_{\xi^j}'')$. We assert that the analytic set Σ is covered by the cones C_j . Let z be any point in $\Sigma \setminus \{0\}$. It is easy to deduce the existence of $s \in \mathbb{C}^*$ such that $s^\beta * z$ lies in K ; and so there exists an index $1 \leq j \leq N$ such that $s^\beta * z$ also lies in C_j . We may suppose that the first entry $|\xi_1^j| \geq 1$, and that Π_{ξ^j} is given as in (10). Hence, there is a pair (t, x) in the Cartesian product $\mathbb{C}^* \times U_{\xi^j}''$ with

$$\begin{aligned} s^\beta * z &= \Pi_{\xi^j}(t, x) = t^\beta * (\xi_1^j, \pi_{\xi^j}(x)); \quad \text{and so} \\ z &= (t/s)^\beta * (\xi_1^j, \pi_{\xi^j}(x)) = \Pi_{\xi^j}(t/s, x). \end{aligned}$$

Previous identity shows that the whole analytic set Σ is covered by the N generalized cones C_1, \dots, C_N . On the other hand, in order to prove the Hölder continuity of (5), we take a fixed parameter $0 < \theta < 1$ and a pair

of points z and w in the intersection of Σ with the open ball B_R of radius $R > 0$ and center at the origin in \mathbb{C}^n . We want to show that there is a constant $C_\Sigma(\theta) > 0$ which does not depend on z or w such that:

$$(14) \quad |g(z) - g(w)| \leq C_\Sigma(\theta) \cdot \text{dist}_\Sigma(z, w)^\theta \cdot \|\lambda\|_\infty.$$

One first step is to show that we only need to verify previous Hölder inequality when the points z and w are both contained in $B_R \cap C_j$, where C_j is a unique generalized cone defined as in the paragraphs above. Let $\epsilon > 0$ be a given parameter. The definition of $\text{dist}_\Sigma(z, w)$ implies the existence of a piecewise smooth curve $\gamma_\epsilon : [0, 1] \rightarrow \Sigma$ joining z and w , i.e. $\gamma_\epsilon(0) = z$ and $\gamma_\epsilon(1) = w$, such that:

$$\text{length}(\gamma_\epsilon) = \int_0^1 \|\gamma'_\epsilon(t)\| dt \leq \text{dist}_\Sigma(z, w) + \epsilon.$$

The image of γ_ϵ is completely contained in $B_R \cap \Sigma$ because Σ is homogeneous (a cone). Now then, we are done if the points z and w are both contained in the same generalized cone C_j . Otherwise, we run over the curve γ_ϵ from z to w , and pick up a finite set $\{z_k\}$ inside $\gamma_\epsilon \subset B_R$ such that: the initial point $z_0 = z$, the final point $z_N = w$, two consecutive elements z_j and z_{j+1} lie in the same generalized cone, and three arbitrary elements of $\{z_k\}$ cannot lie in the same generalized cone. In particular, we may also suppose, without loss of generality, that: $z_0 = z$ is in C_1 , the final point $z_N = w$ is in C_N , and any other point z_j is in the intersection $C_j \cap C_{j+1}$ for every index $1 \leq j < N$. So that, two consecutive points z_{j-1} and z_j lie in the same generalized cone $C_j \cap B_R$ for each index $1 \leq j \leq N$. Assume for the moment that there exist constants $C_\Sigma^j(\theta) > 0$ such that

$$|g(z_{j-1}) - g(z_j)| \leq C_\Sigma^j(\theta) \cdot \text{dist}_\Sigma(z_{j-1}, z_j)^\theta \cdot \|\lambda\|_\infty,$$

for all $1 \leq j \leq N$. Then, it follows that

$$\begin{aligned} |g(z) - g(w)| &\leq \sum_{j=1}^N |g(z_{j-1}) - g(z_j)| \leq \sum_{j=1}^N C_\Sigma^j(\theta) \text{dist}_\Sigma(z_{j-1}, z_j)^\theta \|\lambda\|_\infty \\ &\leq C_\Sigma(\theta) \cdot [\text{dist}_\Sigma(z, w) + \epsilon]^\theta \cdot \|\lambda\|_\infty, \end{aligned}$$

where we have chosen $C_\Sigma(\theta) = \sum_j C_\Sigma^j(\theta)$. Since previous inequality holds for all $\epsilon > 0$, it follows that we only need to prove that the Hölder estimates (14) holds under the assumption that z and w are both contained the intersection of a unique generalized cone C_j with the open ball B_R of radius $R > 0$ and center at the origin in \mathbb{C}^n . Moreover, we can suppose, without loss of generality, that C_j is indeed the generalized cone given in (10).

Recall the given hypotheses: The subvariety Σ is homogeneous (a cone) and has got only one isolated singularity at the origin of \mathbb{C}^n , so that each entry $\beta_k = 1$ in definition 1. We fix a point ξ in the link $K \subset \Sigma$, and assume that its first entry $|\xi_1| \geq 1$. The subvariety Y is then given in (8), and the biholomorphism π is defined from an open set $U \subset \mathbb{C}^m$ into a neighborhood of (ξ_2, \dots, ξ_n) in Y . Let λ be a $(0,1)$ -form as in the hypotheses of Theorem 2. We may easily calculate the pull-back $\Pi^*\lambda$, with the mapping Π given in (10) for all $s \in \mathbb{C}$ and $x \in U$,

$$(15) \quad \Pi(s, x) = s^{(1, \dots, 1)} * (\xi_1, \pi(x)) = (s\xi_1, s\pi(x)) \in \Sigma.$$

The pull-back $\Pi^*\lambda = F_0(s, x)d\bar{s} + \sum_j F_j d\bar{x}_j$ satisfies:

$$\begin{aligned} F_0(s, x) &= \sum_{k=1}^n f_k(\Pi(s, x)) \overline{\pi_k(x)}, \quad \pi_1(x) \equiv \xi_1, \\ F_j(s, x) &= \sum_{k=2}^n f_k(\Pi(s, x)) \overline{\left[s \frac{\partial \pi_k}{\partial x_j} \right]}. \end{aligned}$$

The hypotheses of Theorem 3 yield that the support of every f_k is contain in a ball of radius $R > 0$ and center at the origin. Whence, equation (15) and the fact that $|\xi_1| \geq 1$ automatically imply that each function $F_k(s, x)$ vanishes whenever $|s| > R$. Now then, we restrict the domain of Π to a smaller set $\mathbb{C} \times U''$, where: $U'' \Subset U' \Subset U$, the open set U' is smoothly bounded, and $\pi(U'')$ is a convex neighborhood of (ξ_2, \dots, ξ_n) in Y .

Equation (15) yields that $\Pi_\xi(\{s\} \times U''_\xi)$ is convex for all $s \in \mathbb{C}$ as well. The biholomorphism π has also got a Jacobian (determinant) which is bounded from above and below (away from zero) in the compact closure $\overline{U'_\xi}$. Whence, there exists a constant $D_1 > 0$, such that the following identities hold for every point (s, x) in $\mathbb{C} \times \overline{U'}$ and each index $1 \leq j \leq m$,

$$(16) \quad |F_0(s, x)| \leq D_1 \cdot \|\lambda\|_\infty,$$

$$(17) \quad |F_j(s, x)| \leq D_1 \cdot |s| \cdot \|\lambda\|_\infty.$$

We may show that the Hölder estimate (14) holds for all points z and w in the intersection of the generalized cone $\Pi(\mathbb{C} \times U'')$ with the ball B_R , and so being able to conclude that the same estimate holds on $B_R \cap \Sigma$. Fix the parameter $0 < \theta < 1$. We are going to analyze two different cases. Firstly, we assume there exist a point $x \in U''$ and two complex numbers s and s' , such that $z = \Pi(s, x)$ and $w = \Pi(s', x)$. We say, in this case, that z and w lie in the same complex line. Equation (15) and the fact that $|\xi_1| \geq 1$ yields

that $|s|$ is bounded:

$$(18) \quad |s| \leq |s\xi_1| \leq \|z\| < R.$$

The function $G = g(\Pi)$ defined in (11) and (13) satisfies:

$$\begin{aligned} |g(z) - g(w)| &= |G(s, x) - G(s', x)| \\ &= \frac{1}{2\pi} \left| \int_{|u| \leq R} F_0(u, x) \left(\frac{1}{u-s} - \frac{1}{u-s'} \right) du \wedge d\bar{u} \right|. \end{aligned}$$

Recall that $F_0(u, x)$ vanishes whenever $|u| > R$. It is well known that there exists a constant $D_2(R, \theta) > 0$, depending only on the radius $R > 0$ and the parameter θ , such that:

$$(19) \quad |g(z) - g(w)| \leq D_2(R, \theta) |s - s'|^\theta D_1 \|\lambda\|_\infty.$$

Notice that we have used (16), and consider chapter 6.1 of [12] for a (more general) version of the inequality above. The analysis done in the previous paragraphs shows that (14) holds in the first case. Besides, since both $g(0)$ and $\Pi(0, x)$ vanishes, we also obtain the following useful estimate:

$$(20) \quad |G(s, x)| = |g(z)| \leq D_2(R, \theta) D_1 |s|^\theta \|\lambda\|_\infty.$$

We analyze now the symmetrical case. Let z and \hat{w} be a pair of points in the intersection of $\Pi(\mathbb{C} \times U'')$ with the ball B_R . Assume there exist a complex number $s \neq 0$ and a pair of points x and x' in the open set U'' such that $z = \Pi(s, x)$ and $\hat{w} = \Pi(s, x')$. We say, in this case, that z and \hat{w} lie in the same slice. By a unitary change of coordinates which does not spoil the inequality (17), we may assume that the entries of x and x' are all equal, with the possible exception of the first one. That is, we may assume that both x and x' lie in the complex line $L := \mathbb{C} \times \{(x_2, \dots, x_m)\}$. Recall that the differential $\bar{\partial}G$ is equal to $\Pi^*\lambda$ in the open set $\mathbb{C} \times U$, according to equation (13) and the statement just above it. Hence, we can evaluate $g(z)$ via the inhomogeneous Cauchy-Pompeiu formula on the line L ,

$$\begin{aligned} g(z) = G(s, x) &= \frac{1}{2\pi i} \int_{L \cap U'} F_1(s, t, x_2, \dots, x_m) \frac{dt \wedge d\bar{t}}{t - x_1} \\ &\quad + \frac{1}{2\pi i} \int_{L \cap bU'} G(s, t, x_2, \dots, x_m) \frac{dt}{t - x_1}, \end{aligned}$$

because x is in $L \cap U''$ and $U'' \Subset U'$. We introduce some notation in order to simplify the analysis. The symbols $I_1(s, x)$ and $I_2(s, x)$ stands for the

above integrals on the set $L \cap U'$ and the boundary $L \cap bU'$, respectively. In particular, we have that:

$$g(\hat{w}) = G(s, x') = I_1(s, x') + I_2(s, x').$$

Recall that x and x' are both in $L \cap U''$, and that the difference $x - x'$ is equal to the vector $(x_1 - x'_1, 0, \dots, 0)$. Inequality (17) implies the existence of a constant $D_3(\theta) > 0$, depending only on the diameter of U' and the parameter θ , such that:

$$(21) \quad |I_1(s, x) - I_1(s, x')| \leq D_3(\theta) |x_1 - x'_1|^\theta D_1 |s| \|\lambda\|_\infty.$$

We can calculate similar estimates for I_2 . Let $\delta > 0$ be the distance between the compact sets $\overline{U''}$ and bU' in \mathbb{C}^m . We obviously have that $\delta > 0$ because $U'' \Subset U'$. The following estimates are deduced from (20) and the mean value Theorem, the maximum is calculated over all u in $L \cap U'$,

$$\begin{aligned} |I_2(s, x) - I_2(s, x')| &\leq \frac{|x_1 - x'_1|}{2\pi} \max_v \left| \int_{L \cap bU'} \frac{G(s, t, x_2, \dots) dt}{(t - v)^2} \right| \\ &\leq \frac{|x_1 - x'_1|}{2\pi} \cdot \frac{\text{length}(L \cap bU')}{\delta^2} D_2(R, \theta) D_1 |s|^\theta \|\lambda\|_\infty. \end{aligned}$$

Previous estimates and the inequalities (19) and (21) can be summarized in the following lemma. It is convenient to recall that the points x and x' are both contained in the bounded set $U'' \Subset \mathbb{C}^m$. Moreover, we also have that $|s| < R$ and $|s'| < R$, because z , w and \hat{w} are all contained in the ball B_R ; recall the proof of (18).

Lemma 5 (Isotropic Estimates). *In the situation of Theorems 2 and 3, consider the functions g and Π given in (3) and (15), respectively, and the bounded open set $U'' \Subset \mathbb{C}^m$ defined in the paragraphs above. Then, for every parameter $0 < \theta < 1$, there is a constant $D_4(R, \theta) > 0$ which does not depend on λ such that the following statements hold for all the points $z = \Pi(s, x)$ and $w = \Pi(s', x')$ in the intersection of $\Pi(\mathbb{C} \times U'')$ with the ball B_R :*

$$|g(z) - g(w)| \leq D_4(R, \theta) |s - s'|^\theta \|\lambda\|_\infty,$$

whenever $x = x'$, i.e. z and w are in the same line; and:

$$|g(z) - g(w)| \leq D_4(R, \theta) \|x - x'\|^\theta |s|^\theta \|\lambda\|_\infty,$$

whenever $s = s'$, i.e. z and w are in the same slice.

It is now easy to prove that the Hölder estimates given in (5) and (14) hold for all of points z and w which fulfill the assumptions of Lemma 5; so that they lie in the intersection of the generalized cone $\Pi(\mathbb{C} \times U'')$ with the

ball B_R . The definition of Π , given in (15), allows us to write down the identities:

$$(22) \quad z = \Pi(s, x) = s(\xi_1, \pi(x)) \quad \text{and} \quad w = \Pi(s', x') = s'(\xi_1, \pi(x')).$$

Fix the point $z' := \Pi(s, x') = s(\xi_1, \pi(x'))$ such that it is in the same line than w and in same the slice than z . We can suppose, without loss of generality, that $z' \in B_R$ because z and w also lie in B_R . Otherwise, if the norm $\|z'\| \geq R$, we only need to use $\Pi(s', x)$ instead. We can easily deduce the following estimate from (22) and the fact that $|\xi_1| \geq 1$,

$$|s - s'| \leq |s\xi_1 - s'\xi_1| \leq \|z - w\| \leq \text{dist}_\Sigma(z, w).$$

Recall that π is a biholomorphism whose Jacobian (determinant) is bounded from above and below (away from zero) in the compact set $\overline{U''}$. Moreover, the image $\pi(U'')$ is also a convex set in Y . Whence, recalling (22), we can deduce the existence of a constant $D_5 > 0$, depending only on π and U'' , such that:

$$\begin{aligned} \frac{|s| \cdot \|x - x'\|}{D_5} &\leq |s| \cdot \|\pi(x) - \pi(x')\| \leq \|z - z'\| \\ &\leq \|z - w\| + \|w - z'\| \leq \|z - w\| + |s - s'| \cdot \|(\xi_1, \pi(x'))\| \\ &\leq \text{dist}_\Sigma(z, w) \cdot [2 + \|\pi(x')\|]. \end{aligned}$$

Thus, there exists a constant $D_6 > 0$, depending only on π and U'' , such that the following identities hold for all the points $z = \Pi(s, x)$ and $w = \Pi(s', x')$ in the intersection of $\Pi(\mathbb{C} \times U'')$ with the ball B_R ,

$$|s - s'| \leq D_6 \cdot \text{dist}_\Sigma(z, w) \quad \text{and} \quad |s| \cdot \|x - x'\| \leq D_6 \cdot \text{dist}_\Sigma(z, w).$$

Recall that z' is in the same line than w and in same the slice than z ; so that Lemma 5 automatically yields that:

$$\begin{aligned} |g(z) - g(w)| &\leq |g(z) - g(z')| + |g(z') - g(w)| \\ &\leq D_4(R, \theta) \left[|s|^\theta \cdot \|x - x'\|^\theta + |s - s'|^\theta \right] \|\lambda\|_\infty \\ &\leq D_4(R, \theta) D_6^\theta \text{dist}_\Sigma(z, w)^\theta \|\lambda\|_\infty. \end{aligned}$$

This completes the proof that the Hölder estimates given in (5) and (14) hold for all of points z and w in the intersection of the ball B_R with the generalized cone $\Pi(\mathbb{C} \times U'')$; and so we can conclude that the same Hölder estimates hold for all point z and w in $B_R \cap \Sigma$, we just need to recall the analysis done in the paragraphs located between equations (14) and (15).

4. L^2 -ESTIMATES

We prove Theorem 4 in this section, so we begin by showing that the function g given in (6) is indeed well defined. Recall that Σ is a pure d -dimensional homogeneous (cone) subvariety of \mathbb{C}^n , so that $n \geq 2$ and each entry $\beta_k = 1$ in Definition 1. Besides, consider the differential form λ given by $\sum_k f_k d\bar{z}_k$, where the coefficients f_k are all square-integrable functions in Σ . Assume that the support of λ is contained in the open ball B_R of radius $R > 0$ and center at the origin. We only need to show that the following integrals exist,

$$(23) \quad \int_{w \in \mathbb{C}} \int_{z \in \Sigma \cap B_R} \left| \frac{f_k(wz)w^d z_k}{w(w-1)} \right| dV_\Sigma dV_{\mathbb{C}} < \infty.$$

A direct application of Fubini's theorem will yield that the integrals in (6) are all well defined for almost all z in $\Sigma \cap B_R$, and so they are also well defined for almost all $z \in \Sigma$ because the radius R can be as large as we want. The fact that Σ is a pure $2d$ -real dimensional and homogeneous (cone) subvariety automatically implies the existence of a constant $C_0 > 0$ such that the following equations hold for all $w \in \mathbb{C}$ and real $\rho > 0$:

$$(24) \quad \int_{z \in \Sigma \cap B_\rho} \|z\|^2 = C_0 \rho^{2d+2}, \quad \int_{z \in \Sigma} |f_k(wz)|^2 = \frac{\|f_k\|_{L^2(\Sigma)}^2}{|w|^{2d}}.$$

Notice that we need not calculate the integral (23) in the Cartesian product of \mathbb{C} times $\Sigma \cap B_R$. We can simplify the calculations by integrating over the set Ξ defined below, because $f_k(wz) = 0$ whenever $\|wz\| \geq R$,

$$(25) \quad \Xi := \{(w, z) \in \mathbb{C} \times \Sigma : \|z\| < R, \|wz\| < R\}.$$

We easily have that:

$$(26) \quad \begin{aligned} \left\| \frac{f_k(wz)w^d}{|w^2 - w|^{2/3}} \right\|_{L^2(\Xi)}^2 &\leq \int_{w \in \mathbb{C}} \int_{z \in \Sigma} \frac{|f_k(wz)w^d|^2}{|w^2 - w|^{4/3}} \\ &\leq \|f_k\|_{L^2(\Sigma)}^2 \int_{w \in \mathbb{C}} \frac{1}{|w^2 - w|^{4/3}} < \infty; \end{aligned}$$

and that:

$$(27) \quad \begin{aligned} \left\| \frac{z_k}{|w^2 - w|^{1/3}} \right\|_{L^2(\Xi)}^2 &\leq \int_{w \in \mathbb{C}} \int_{\substack{z \in \Sigma \cap B_R, \\ \|z\| < R/|w|}} \frac{\|z\|^2}{|w^2 - w|^{2/3}} \\ &\leq \int_{|w| \leq 1} \frac{C_0 R^{2d+2}}{|w^2 - w|^{2/3}} + \int_{|w| > 1} \frac{C_0 (R/|w|)^{2d+2}}{|w^2 - w|^{2/3}} < \infty. \end{aligned}$$

The last integral in the first line of (27) must be separated into two parts according to the fact that $|w|$ is either less or greater than one, and then,

one must apply (24) with ρ respectively equal to R or $R/|w|$. The Cauchy-Schwartz inequality $\|ab\|_{L^1} \leq \|a\|_{L^2}\|b\|_{L^2}$ allows us to deduce equation (23) from the inequalities (26)–(27), for we only need to integrate on the set Ξ given in (25). Thus, a direct application of Fubini's theorem implies that the following function given in (6) is well defined for almost all $z \in \Sigma$,

$$(28) \quad g(z) = \sum_{k=1}^n \frac{H_k(z)}{\pi}, \quad H_k(z) = \int_{|w| \leq \frac{R}{\|z\|}} f_k(wz) \frac{w^d \overline{z_k} dw \wedge d\overline{w}}{w(w-1)2i}.$$

Notice that $f_k(wz) = 0$ whenever $\|wz\| \geq R$, because of the given hypotheses. Moreover, the L^2 -estimate (7) easily follows from the following inequalities (Cauchy-Schwartz) and equations (26)–(27):

$$\begin{aligned} |H_k(z)|^2 &\leq \int_{|w| \leq \frac{R}{\|z\|}} \frac{|f_k(wz)w^d|^2}{|w^2 - w|^{4/3}} \int_{|\omega| \leq \frac{R}{\|z\|}} \frac{|z_k|^2}{|\omega^2 - \omega|^{2/3}} \quad \text{and} \\ \int_{z \in \Sigma \cap B_R} |H_k(z)|^2 &\leq \left\| \frac{f_k(wz)w^d}{|w^2 - w|^{2/3}} \right\|_{L^2(\Xi)}^2 \left\| \frac{z_k}{|w^2 - w|^{1/3}} \right\|_{L^2(\Xi)}^2. \end{aligned}$$

Recall that $|\frac{dw \wedge d\overline{w}}{2i}|$ is the volume differential in \mathbb{C} , that $\|f_k\|_{L^2(\Sigma)}$ is less than equal to $\|\lambda\|_{L^2_{0,1}(\Sigma)}$, and that we only need to integrate on the set Ξ given in (25). Finally, we prove that g in (28) satisfies the differential equation $\overline{\partial}g = \lambda$. We only need to follow step by the step the proof presented in Section 2. The only difference is that we must use a weighted Cauchy-Pompeiu integral in (13), with $m = d-1$ integer:

$$(29) \quad \mathcal{G}(s, x) := \frac{1}{2\pi i} \cdot \frac{1}{s^m} \int_{u \in \mathbb{C}} \frac{u^m F_0(u, x)}{u - s} du \wedge d\overline{u},$$

$$\text{where} \quad F_0(u, x) = \sum_{k=1}^n f_k(\Pi(u, x)) \overline{\pi_k(x)}.$$

Notice that that $\Pi(u, x) = u(\xi_1, \pi(x))$ because each entry $\beta_k = 1$ in (2) and (10). We obviously have that $\overline{\partial}\mathcal{G} = [s^m \Pi^* \lambda]/s^m$. Hence, the function g given in (28) is a solution to $\overline{\partial}g = \lambda$, because $g(\Pi(s, x))$ is identically equal to (29) after setting $u = sw$ and $\pi_1(x) \equiv \xi_1$. This concludes the proof of Theorem 4.

5. WEIGHTED HOMOGENEOUS ESTIMATES

We want to close this paper presenting a useful technique for generalizing the estimates given in Theorems 3 and 4, so as to consider weighted homogeneous subvarieties instead of cones. Let $X \subset \mathbb{C}^n$ be a weighted homogeneous subvariety with only one singularity at the origin and defined as the zero

locus of a finite set of polynomials $\{Q_k\}$. Thus, the polynomials $Q_k(x)$ are all weighted homogeneous with respect to the same vector $\beta \in \mathbb{Z}^n$, and each entry $\beta_k \geq 1$. Define the following holomorphic mapping:

$$(30) \quad \Theta : \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad \text{with} \quad \Theta(z) = (z_1^{\beta_1}, z_2^{\beta_2}, \dots, z_n^{\beta_n}).$$

It is easy to see that each polynomial $Q_k(\Theta)$ is homogeneous, and so the subvariety $\Sigma \subset \mathbb{C}^n$ defined as the zero locus of $\{Q_k(\Theta)\}$ is a cone. Moreover, since Θ is locally a biholomorphism in $\mathbb{C}^n \setminus \{0\}$, we have that Σ has got only one singularity at the origin as well. Consider a $(0,1)$ -form \aleph given by the sum $\sum_k f_k d\bar{x}_k$, where the coefficients f_k are all Borel-measurable functions in X , and x_1, \dots, x_n are the Cartesian coordinates of \mathbb{C}^n . We may follow two different paths in order to solve the equation $\bar{\partial}h = \aleph$. We may apply the main Theorem 2, whenever \aleph is bounded and has compact support on X , so as to get the solution:

$$h(x) = \sum_{k=1}^n \frac{\beta_k}{2\pi i} \int_{w \in \mathbb{C}} f_k(w^\beta * x) \frac{(\overline{w^{\beta_k} x_k}) dw \wedge d\bar{w}}{\bar{w} (w-1)}.$$

Otherwise, we may consider the pull-back $\Theta^*\aleph$, and apply Theorem 3, in order to solve the equation $\bar{\partial}g = \Theta^*\aleph$ on Σ . We easily have that:

$$\begin{aligned} \Theta^*\aleph &= \sum_{k=1}^n f_k(\Theta(z)) \beta_k \bar{z}_k^{\beta_k-1} d\bar{z}_k \quad \text{and} \\ g(z) &= \sum_{k=1}^n \frac{\beta_k}{2\pi i} \int_{w \in \mathbb{C}} f_k(\Theta(wz)) \frac{(\overline{wz_k})^{\beta_k} dw \wedge d\bar{w}}{\bar{w} (w-1)}. \end{aligned}$$

Both paths yield exactly the same solution because $g(z)$ is identically equal to $h(\Theta(z))$. Recall that $w^\beta * \Theta(z)$ is equal to $\Theta(wz)$ for all $w \in \mathbb{C}$ and $z \in \mathbb{C}^n$. Hence, we may calculate the solution $g(z)$ above, and use the Hölder estimates given in equation (5),

$$|g(z) - g(w)| \leq C_\Sigma(R, \theta) \cdot \text{dist}_\Sigma(z, w)^\theta \cdot \|\Theta^*\aleph\|_\infty.$$

A final steep is to *push forward* these estimates, in order to deduce similar Hölder estimates for the solution $h(x)$ on X . A detailed analysis on the procedure for *pushing forward* the Hölder estimates can be found in [14]. On the other hand, we may use a similar procedure for L^2 -estimates. In that case the subvariety X can have arbitrary singularities.

REFERENCES

- [1] F. Acosta; E. S. Zeron. *Hölder estimates for the $\bar{\partial}$ -equation on surfaces with simple singularities*. Bol. Soc. Mat. Mexicana (3). **12** (2006), no. 2, pp. 193–204.

- [2] F. Acosta; E. S. Zeron. *Hölder estimates for the $\bar{\partial}$ -equation on surfaces with singularities of the type E_6 and E_7* . Bol. Soc. Mat. Mexicana (3). **13** (2007), no. 1.
- [3] K. Diederich; J. E. Fornæss; S. Vassiliadou. *Local L^2 results for $\bar{\partial}$ on a singular surface*. Math. Scand. **92** (2003), pp. 269–294.
- [4] A. Dimca. *Singularities and topology of hypersurfaces*, (Universitext). Springer-Verlag, New York, 1992.
- [5] J. E. Fornæss. *L^2 results for $\bar{\partial}$ in a conic*. International Symposium, Complex Analysis and Related Topics (Cuernavaca, 1996). pp. 67–72. Oper. Theory Adv. Appl., 114. Birkhauser, Basel, 2000.
- [6] J. E. Fornæss; E. A. Gavosto. *The Cauchy Riemann equation on singular spaces*. Duke Math. J. **93** (1998), no. 3, pp. 453–477.
- [7] J. E. Fornæss; N. Øvrelid, S. Vassiliadou. *Semiglobal results for $\bar{\partial}$ on a complex space with arbitrary singularities*. Proc. Amer. Math. Soc. **133** (2005), no. 8, pp. 2377–2386.
- [8] J. E. Fornæss; N. Øvrelid; S. Vassiliadou. *Local L^2 results for $\bar{\partial}$: the isolated singularities case*. Internat. J. Math. **16** (2005), no. 4, pp. 387–418.
- [9] E. A. Gavosto. *Hölder estimates for the $\bar{\partial}$ -equation in some domains of finite type*. J. Geom. Anal. **7** (1997), no. 4, pp. 593–609.
- [10] G. Henkin; J. Leiterer. *Andreotti-Grauert theory by integral formulas*, (Progress in Mathematics, 74). Birkhäuser-Verlag, Basel, 1988.
- [11] I. Lieb; J. Michel. *The Cauchy-Riemann Complex, integral formulae and Neumann problem*, (Aspects of Mathematics, E34). Vieweg, Braunschweig, 2002.
- [12] J. Ruppenthal. *Zur Regularität der Cauchy-Riemannschen Differentialgleichungen auf komplexen Räumen*. Bonner Math. Schr. **380** (2006).
- [13] G. Scheja. *Riemannsche Hebbarkeitssätze für Cohomologieklassen*. Math. Ann. **144** (1961), pp. 345–360.
- [14] M. Solís; E. S. Zeron. *Hölder estimates for the $\bar{\partial}$ -equation on singular quotient varieties*. Bol. Soc. Mat. Mexicana (3). **In press**.
- [15] N. Øvrelid; S. Vassiliadou. *Solving $\bar{\partial}$ on product singularities*. Complex Var. Elliptic Equ. **51** (2006), no. 3, pp. 225–237.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, 530 CURCH STREET,
ANN ARBOR, MI 48109, USA.

E-mail address: jean@math.uni-bonn.de

DEPTO. MATEMÁTICAS, CINVESTAV, APARTADO POSTAL 14-740, MÉXICO D.F.,
07000, MÉXICO.

E-mail address: eszeron@math.cinvestav.mx